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Finite Topologies and Hamiltonian Paths

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Suppose X is a finite set. This paper deals with the question of how many mutually complementary topologies X can carry. If p is a prime and $|X| = p$, $p+1$, $2p-1$ or $2p$, we prove that the answers are respectively p, p, $2p-1$, $2p-1$. The problem is shown to be related to the existence of a certain type of 1-factorization of the complete graph on an even number of points, and is also formulated combinatorially.

1. INTRODUCTION

Suppose X is a nonempty set and Ψ is a collection such that (1) F in Ψ implies $F \subset 2^X$, ϕ, X belong to F and F is closed under arbitrary unions and finite intersections and if F and G are distinct members of Ψ, $F = \{f_i : i \in I\}$ and $G = \{g_j : j \in J\}$, then (2) for each x in X, there exists an i in I and j in J such that $f_i \cap g_j = \{x\}$ and (3) if $f_k = g_l$, then $f_k = \phi$ or $f_k = X$. The basic question posed in this note is that of finding the sup of the set of cardinals d such that X carries a collection Ψ of cardinal d. In the parlance of topology, which is where this problem arose, Ψ is a collection of mutually complementary topologies. Topologies σ and τ on X are said to be complementary if their sup is the discrete topology and their inf is the indiscrete topology. This problem was studied in [1–3] where only the case $|X| \geqslant \aleph_0$ was considered. In [3] it was shown that if $|X| \geqslant \aleph_0$, then X carries a collection Ψ such that $|\Psi| = |X|$ and Ψ has the above properties.

If $|X| < \aleph_0$, there is a combinatorial problem which is often (perhaps always) equivalent to the above question and which may be of independent

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interest. We shall save a general formulation of this combinatorial question until later. A sample is this. If $|X| = 10$, is it possible to split the 45 pairs of elements of X into nine groups of five pairs each such that every element of X belongs to some pair in each group and X itself is the only set that can be obtained as a union of pairs from two different groups? We shall see that the answer in this case is affirmative.

The existence theorems will be proved using the techniques of modular arithmetic. We shall make no attempt to state and prove everything using the terminology of a single field, but rather shall purposely employ terminology from the several fields involved in the hope that the various ways of looking at the problem will be helpful in obtaining a complete solution. Theorem 1 with its elegant proof is due to J. B. Kelly.

2. Reductions and Reformulations

Definition. Suppose n is a positive integer, $n \geqslant 2$, and X is a set such that $|X| = n$. Then F_n is the largest positive integer m such that X carries a family of m mutually complementary topologies.

Proposition 1. $F_n \leqslant n$.

Proof. Suppose $\tau_1, \ldots, \tau_j$ is a family of mutually complementary topologies on X. If $a \in X$ and $U_i(a)$ is the minimal element of τ_i containing a, then $1 \leqslant i, k \leqslant j$ and $i \neq k$ implies $U_i(a) \cap U_k(a) = \{a\}$ and $U_i(a) \neq U_k(a)$. An application of the pigeonhole principle shows that $j \leqslant n$. Note that if $j = n$, and $X = \{1, 2, \ldots, n\}$, then at every i in X, the minimal open sets of $\tau_1, \ldots, \tau_n$ must be a permutation of $\{i\}$, $\{i, 1\}, \ldots, \{i, i-1\}, \{i, i+1\}, \ldots, \{i, n\}$.

Proposition 2. *If $n \geqslant 2$, then $F_{2n} \leqslant (2n - 1)$.*

Proof. We know that $F_{2n} \leqslant 2n$. Suppose $\tau_1, \ldots, \tau_{2n}$ is a family of $2n$ mutually complementary topologies on X, where $X = \{1, \ldots, 2n\}$. Then, for each i, $1 \leqslant i \leqslant 2n$, the minimal open sets about i in the $2n$ topologies have the form described above. If τ_k isolates i and j, then the pair $\{i, j\}$ must occur as the minimal open set about i in some τ_p and about j in some τ_q, where p and q might be equal. In such a case τ_k and τ_p have a nontrivial open set in common; namely $\{i, j\}$, and are not complements. Thus, each τ_k must isolate exactly one point.

We may assume the labeling is such that τ_1 isolates 1. The minimal τ_1-open set containing 2 must be a doubleton. If it is $\{1, 2\}$, then τ_1 has this open set in common with the topology τ_k, $k \neq 1$, where the minimal

τ_k-open set about 1 is $\{1, 2\}$. Thus, we may assume the labeling is such that the minimal τ_1-open set containing 2 is $\{2, 3\}$. Clearly this must also be the minimal τ_1-open set containing 3. Continue this process and we see that the minimal τ_1-open set containing $2n$ is $\{1, 2n\}$ and hence τ_1 has this open set in common with the topology τ_l, $l \neq 1$, where the minimal τ_l-open set about 1 is $\{1, 2n\}$. Thus, the assumption that $F_{2n} = 2n$ leads to a contradiction.

We note in passing that the pigeonhole technique of Proposition 1 also proves that if X is infinite and $\prod$ is the lattice of principal topologies on X, then $|X|$ is the largest cardinal d such that X carries a family of d mutually complementary principal topologies. In the terminology of [3], $|X| = w^*(\prod)$.

Definition. Suppose n is a positive integer, $n \geqslant 2$. If X is a set such that $|X| = 2n - 1$, then G_{2n-1} is the largest positive integer m such that X carries a family of m mutually complementary topologies, each of which has the property that its minimal open sets are all doubletons, except for one singleton.

If X is a set such that $|X| = 2n$, then G_{2n} is the largest positive integer m such that X carries a family of m mutually complementary topologies, each of which has the property that its minimal open sets are all doubletons.

Clearly, for each positive integer n, $n \geqslant 3$, $G_n \leqslant F_n$. We will show that in a large number of cases $G_n = F_n$ and this number is the maximum allowed by Propositions 1 and 2.

In order to prove this and other facts as well, it is very helpful to reformulate the definitions of G_{2n-1} and G_{2n} in graph-theoretic and combinatorial terms.

Definition. We call the topologies of the sort described in the definition of G_{2n-1} and G_{2n}, Hamiltonian topologies.

It is clear that in the even case, these topologies can be thought of as 1-factors of the complete graph on X [4, p. 84].

The reason for this terminology is easy to explain. Let us consider the odd case first. Suppose $\tau_1, \ldots, \tau_r$ is a family of mutually complementary Hamiltonian topologies on X, where $|X| = 2n - 1$. If τ_i and τ_j are distinct members of this family, then each isolates exactly one point, and the isolated points are distinct. The doubletons in τ_i and τ_j can be considered as forming a graph on the set X. It is apparent that if we start at the point isolated by τ_i and alternate between lines contributed by τ_j and τ_i, we arrive at the point isolated by τ_j only after having traveled a

Hamiltonian path through X. In the even case, we may start at any point and the corresponding technique will lead to a Hamiltonian circuit through X. The fact that we get a Hamiltonian path or circuit means, topologically, that the inf of the two topologies is the indiscrete topology.

Thus, the problem of determining $G_{2n-1}(G_{2n})$ can be pictured as that of finding Hamiltonian topologies that mesh together properly to form Hamiltonian paths (circuits).

PROPOSITION 3. *If n is a positive integer, $n \geqslant 2$, then $G_{2n-1} = G_{2n}$.*

Proof. Suppose $\tau_1, \ldots, \tau_p$ is a family of G_{2n-1} mutually complementary Hamiltonian topologies on $X = \{1, 2, \ldots, 2n-1\}$. Suppose $X^* = X \cup \{2n\}$, and modify each τ_i to τ_i^* by pairing the τ_i-isolated point with $2n$. The new topologies are Hamiltonian on X^*. It is obvious that the sup of any two of them is discrete and since any two of them lead to a Hamiltonian circuit of X^*, the inf of any two is indiscrete. Thus, $G_{2n-1} \leqslant G_{2n}$.

Going the other way, we take away a point and isolate the points paired with it. This gives $G_{2n} \leqslant G_{2n-1}$.

Finally, we reformulate the definitions of G_{2n-1} and G_{2n} in combinatorial terms. Suppose we have $(2n-1)$ objects, $n \geqslant 2$. Then G_{2n-1} is the largest number of ways one can split the objects into groups, each group consisting of n pairwise disjoint sets, one singleton and $(n-1)$ doubletons, such that no proper subset of the $(2n-1)$ objects can be expressed in more than one way as a union of sets of a group. Similarly G_{2n} is the largest number of ways one can split $2n$ objects into groups, each group consisting of n pairwise disjoint doubleton sets, such that no proper subset of the $2n$ objects can be expressed in more than one way as a union of sets of a group.

3. DETERMINATION OF F_n AND G_n

THEOREM 1. *If p is an odd prime, then $F_p = G_p = p$.*

Proof. Suppose $X = \{0, 1, \ldots, p-1\}$. The group with singleton $\{a\}$ is defined to include all pairs $\{x, y\}$ such that $x + y \equiv 2a \pmod{p}$. Then, the sum of the elements in any set union from this group is congruent to some $ka \pmod{p}$. If a proper subset of X can be expressed as a union of sets from two different groups, say groups which have the singletons $\{a\}$ and $\{b\}$, then we have for some k, $1 \leqslant k \leqslant p-1$, that $ka \equiv kb \pmod{p}$ and hence $a \equiv b \pmod{p}$, a contradiction.

COROLLARY 1. *If p is an odd prime, then $F_{p+1} = G_{p+1} = p$.*

Proof. This is an immediate consequence of the preceding Theorem and Proposition 3.

It is clear that $F_2 = 2$. The argument of Theorem 1 can be modified to prove the following.

THEOREM 2. *If p is an odd prime, then $F_{2p} = G_{2p} = 2p - 1$.*

Proof. Consider two copies of the first p nonnegative integers, say $X = \{0, 1, \ldots, p - 1\}$ and $Y = \{0^*, 1^*, \ldots, (p - 1)^*\}$. Apply the technique of Theorem 1 to find p groups on X and p groups on Y. We shall sew X-groups to Y-groups by pairing isolated points as follows. Pair a and b^* iff $a + b^* = p - 1$. For example, the X-group that isolates 0 is paired with the Y-group that isolates $(p - 1)^*$ to form a group on $X \cup Y$ consisting of the doubletons in the two groups and the new doubleton $\{0, (p - 1)^*\}$. Similarly, the X-group that isolates $(p - 1)$ is paired with the Y-group that isolates 0^*. Call these p new groups of doubletons on $X \cup Y$ the "outside" groups. It is clear that if we alternate lines between two of these new groups, we travel a Hamiltonian circuit on $X \cup Y$. Thus, the outside groups form a family of p mutually complementary Hamiltonian topologies on $2p$ points.

We now define $p - 1$ "inside" groups. Note that of the new pairs formed in the outside groups, only $(p - 1)/2$ is paired with itself, so to speak. Furthermore, every pairing from X to Y corresponds to a doubleton in the X-group that isolates $(p - 1)/2$. Consider all the X-groups except the one that isolates $(p - 1)/2$. There are $(p - 1)$ of these groups. Each of these groups defines an inside group as follows. If an X-group isolates i, $i \neq (p - 1)/2$, then the associated inside group consists of $\{i, i^*\}$ and all pairs $\{j, k^*\}$ and $\{k, j^*\}$ such that $\{j, k\}$ is in the X-group that isolates i. These are all new pairings since the sum of the two numbers of an inside group doubleton is congruent to $2a \pmod p$, where $a \neq (p - 1)/2$, whereas the sum of two numbers of an outside group doubleton containing one element from X and one from Y is congruent to $2((p - 1)/2)$ $\pmod p$.

Now, if a proper subset of $X \cup Y$ can be expressed as a union of doubletons from two different inside groups, say the ones associated with a and b, when $a, b \neq (p - 1)/2$, then for some n, $1 \leqslant n \leqslant p - 1$, $2na \equiv 2nb$ $\pmod p$. But $2n < 2p$ implies $(2n, p) = 1$ and thus $a \equiv b \pmod p$, a contradiction. It follows that the inside groups form a family of $(p - 1)$ mutually complementary Hamiltonian topologies on $2p$ points.

The proof will be complete if it can be shown that no proper subset of $X \cup Y$ is the union of doubletons of an inside group and an outside group. Suppose such a proper subset U exists. It is clear that $[(X \cup Y) - U]$

will also have this property. Since inside groups are defined in such a way that every doubleton has one element in X and the other in Y, it is clear that either U or its complement has an even number of points in X and an even number of points in Y. Assume the labeling such that U has $2n$ points on each side, $1 \leqslant n < p/2$. Let $x_1, ..., x_{2n}$ be the points in $U \cap X$. Since $U \cap X$ is a union of doubletons in one of the p groups on X, there is an a, $0 \leqslant a \leqslant p - 1$, such that $\sum x_i \equiv 2na \pmod p$. Then, because of the way we paired isolated points to form the outside groups, if $y_1, ..., y_{2n}$ are the points in $U \cap Y$, $\sum y_i \equiv 2n(p - 1 - a) \pmod p$. Thus,

$$\sum x_i + \sum y_i \equiv 2n(p - 1) \pmod p \equiv 4n(p - 1)/2 \pmod p.$$

But we may assume the labeling is such that x_i pairs with y_i in the inside group. It follows that there is a k, $0 \leqslant k \leqslant p - 1$ and $k \neq (p - 1)/2$, such that $\sum(x_i + y_i) \equiv 4nk \pmod p$. Since $2n < p$, we have $4n < 2p$ so that $(4n, p) = 1$ and $k \equiv (p - 1)/2 \pmod p$, a contradiction.

COROLLARY 2. *If p is an odd prime, then $F_{2p-1} = G_{2p-1} = 2p - 1$.*

Proof. This is an immediate consequence of Theorem 2 and Proposition 3.

The results proved above show that the values of F_n and G_n are, in many cases, the best possible. For example, we have disposed of 67 or the 99 numbers 2 through 100. The first few unsettled cases are the pairs {15, 16}, {27, 28}, {35, 36}.

4. EXAMPLES

If we follow the procedure described in Theorem 2 for the case $p = 5$, and change Y to {5, 6, 7, 8, 9}, we obtain the following groups of pairs.

{0, 5},	{1, 4},	{2, 3},	{6, 9},	{7, 8}
{1, 6},	{0, 2},	{3, 4},	{5, 7},	{8, 9}
{2, 7},	{1, 3},	{0, 4},	{6, 8},	{5, 9}
{3, 8},	{0, 1},	{2, 4},	{5, 6},	{7, 9}
{4, 9},	{0, 3},	{1, 2},	{5, 8},	{6, 7}
{0, 9},	{1, 5},	{4, 8},	{2, 6},	{3, 7}
{1, 8},	{2, 9},	{0, 7},	{3, 5},	{4, 6}
{3, 6},	{2, 5},	{4, 7},	{0, 8},	{1, 9}
{4, 5},	{0, 6},	{3, 9},	{1, 7},	{2, 8}.

The first five groups correspond to the outside groups of Theorem 2 and the last four correspond to inside groups.

It should be clear that things have to be chosen just right to get G_n groups. One can't just arbitrarily try to extend a set of mutually complementary Hamiltonian topologies and hope to reach the maximum value. For example, we know $G_5 = 5$, but here are three mutually complementary Hamiltonian topologies which can't be extended to four mutually complementary Hamiltonian topologies:

$$\{1, 3\}, \{2, 4\}, \{5\}; \quad \{1, 4\}, \{2, 5\}, \{3\}; \quad \{1, 5\}, \{2, 3\}, \{4\}.$$

Note, however, that $\{1, 2\}$, $\{3, 4, 5\}$ is complementary to each of those topologies, although it is not itself a Hamiltonian topology.

It also does not suffice to choose Hamiltonian circuits with no pairs in common, and then split each of these circuits into two Hamiltonian topologies. One easily constructs examples where Hamiltonian topologies from disjoint Hamiltonian circuits don't mesh properly.

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